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Conserved quantities of generalized periodic box–ball systems constructed from the ndKP equation

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Abstract

We investigate periodic box–ball systems (PBBSs) with several kinds of balls and box capacity greater than or equal to one. Conserved quantities of the PBBSs are constructed from those of the nonautonomous discrete KP (ndKP) equation using the Lax representation of the ndKP equation.

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1. Introduction

A cellular automaton (CA) is a discrete dynamical system consisting of regular arrays of cells [1]. Each cell takes only a finite number of states and is updated in discrete time steps. Although the updating rule is simple, CAs often exhibit very complicated time evolution patterns and they have been investigated as good models for natural and/or social phenomena. A box–ball system (BBS) is a filter-type CA which is expressed as a discrete dynamical system of balls in an infinite array of boxes [2, 3]. One of the peculiar features of the BBS is that it is actually an *integrable* CA and this for two reasons. One reason is that a BBS is obtained from an integrable nonlinear equation through a limiting procedure called ultradiscretization [4, 5], and the other reason is that it is regarded as an integrable lattice model at zero temperature [6–8]. Accordingly, the BBS has soliton solutions and a sufficiently large number of conserved quantities [9].

The periodic box–ball system (PBBS) is the BBS in which the updating rule is extended to be compatible with a periodic boundary condition [10]. Let us consider a one-dimensional array of N boxes. A periodic boundary condition is imposed by assuming that the Nth box is adjacent to the first one. (We may imagine that the boxes are arranged in a circle.) The capacity of the nth ($1 \leqslant n \leqslant N$) box is denoted by a positive integer θ_n. We suppose that there are M kinds of balls distinguished by an integer index j ($1 \leqslant j \leqslant M$). Then, the rule

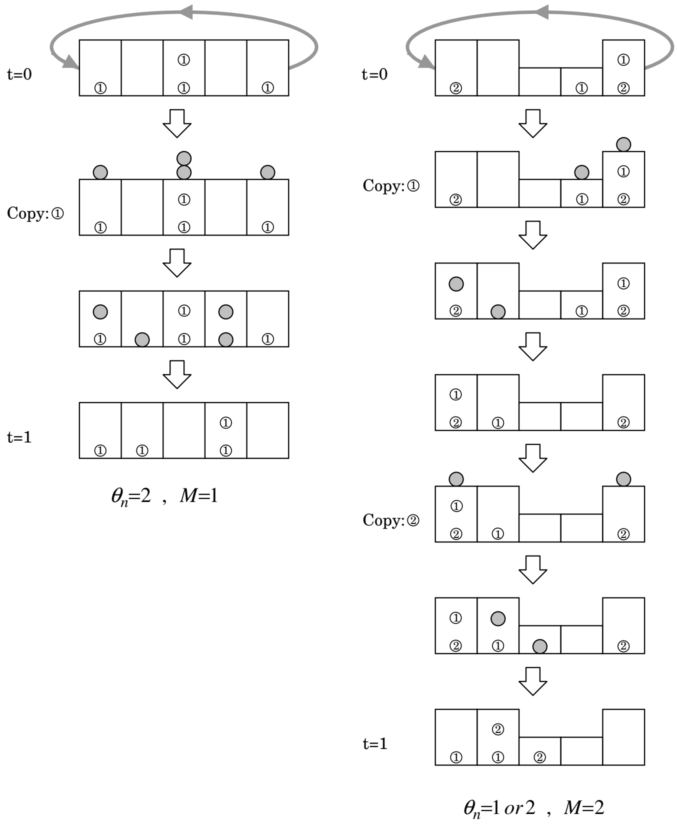

Figure 1. Time evolution rule for PBBS.

for the time evolution from time step t to $t + 1$ is given as follows:

1. At each box, create the same number of copies of the balls with index 1.
2. Choose one of the copies arbitrarily and move it to the nearest box with an available space to the right of it.
3. Choose one of the remaining copies and move it to the nearest available box on the right of it.
4. Repeat the above procedure until all the copies have been moved.
5. Delete all the original balls with index 1.
6. Perform the same procedure for the balls with index 2.
7. Repeat this procedure successively until all of the balls are moved.

An example of the time evolution of the PBBS according to this rule is shown in figure 1.

Since the PBBS is composed of a finite number of boxes and balls, it can only take on a finite number of patterns. Hence its trajectory is always periodic and a fundamental cycle, i.e. the shortest period of the periodic motion, exists for any given initial state.

In the case where the box capacity is one everywhere and only one kind of ball exists, the PBBS is obtained from the discrete Toda equation [11], which is a well-known integrable partial difference equation, with a periodic boundary condition through a limiting procedure. Using inverse ultradiscretization, the initial value problem of the PBBS is solvable by the inverse scattering transform [12], and we can obtain the explicit formulae expressing the fundamental

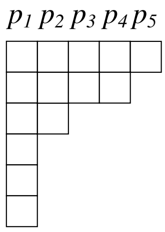

Figure 2. Young diagram corresponding to the conserved quantities of (#).

cycle for a given initial state of the PBBS [13]. Furthermore, using these formulae, we can estimate the asymptotic behaviour of the fundamental cycles which shows an important number theoretical aspect of the PBBS [14, 15]. One of the key elements underlying these results is that we can construct the conserved quantities of the PBBS explicitly. Denoting a vacant box by 0 and a filled box by 1, we obtain the 0, 1 sequence corresponding to a state of the PBBS. (We regard the last entry of the sequence as adjacent to the first entry.) Then the explicit algorithm to construct the conserved quantities is as follows [10, 16].

1. Let p_1 be the number of 10 in the sequence.
2. Eliminate all the 10 in the original sequence and let p_2 be the number of 10 in the new sequence.
3. Repeat the above procedure until all the 1 are eliminated.
4. Then the decreasing positive integer sequence $\{p_1, p_2, p_3, \ldots\}$ consists of the conserved quantities.

For example, for the state

(#) 00 111 011 100 100 011 110 001 101 000 000

we have $p_1 = 6$, and eliminating 10, we obtain a new sequence

0 011 110 001 110 010 000.

and $p_2 = 3$. In a similar manner, we have $p_3 = 2$, $p_4 = 2$, $p_5 = 1$. To see that these $\{p_j\}$ are conserved, we evolve (#) by one time step

(#′) 00 000 100 011 011 100 001 110 010 111 100.

By applying the above algorithm again, we find the same integer sequence $\{p_j\}\,(j = 1, 2, \ldots, 5)$.

Since the sequence $p_1 p_2 \cdots$ is a decreasing positive integer sequence, we can associate a Young diagram to it by regarding p_j as the number of squares in the jth column of the diagram. For example, the Young diagram corresponding to the state (#) is shown in figure 2.

When we denote by L_j the length of the jth rows of the Young diagram, the decreasing integer sequence $\{L_1, L_2, \ldots\}$ is another expression for the conserved quantities of the PBBS. They are sometimes called the lengths of solitons [13], because in the case of an infinite number of boxes (or for the original BBS), after sufficiently large time steps, the state of the PBBS consists of solitons which are arranged according to the order of their lengths and which move freely. We can prove that the length of the jth largest soliton among these freely moving solitons coincides with L_j. Hence, for a given initial state, we can find the solitons which constitute that state after sufficiently many time steps by constructing the corresponding Young diagram. Note that the solitons of the PBBS can be defined for any state as shown in [13]. We shall use this fact in section 7 to prove the correspondence between our results and the previous ones.

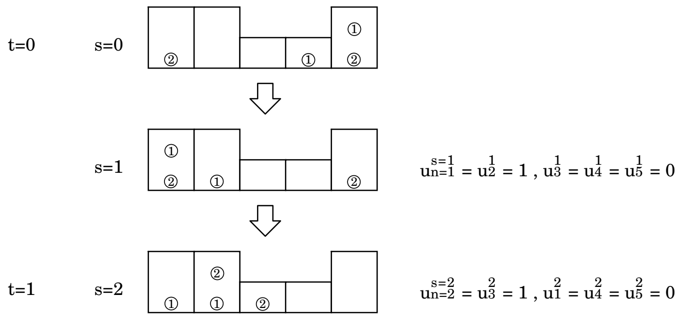

Figure 3. s and u_n^s.

For a general PBBS with box capacity greater than one and several kinds of balls however, we know only a few of its features, in part because we did not yet obtain a formula for the conserved quantities. In this paper, we investigate the conserved quantities of the generalized PBBS. In section 2, we obtain an ultradiscrete equation for the generalized PBBS. In sections 3 and 4, we consider a reduction of the ndKP equation and its Lax representation, and show that its ultradiscrete limit gives the ultradiscrete equation for the PBBS. Using these results, we obtain a formula to calculate the conserved quantities of the PBBS in section 5. In section 6, we treat the case with only one kind of ball but with arbitrary box capacities and give an explicit expression for the conserved quantities, and, in the following section, prove that they coincide with those expressed in terms of a Young diagram when all the box capacities are one. Concluding remarks are given in section 8.

2. Periodic box–ball system

In order to describe the dynamics of the PBBS in more detail, we introduce a new independent variable s ($s \in \mathbb{Z}$). As any integer s can be uniquely expressed as $s = Mt + j$ ($t \in \mathbb{Z}, 1 \leqslant j \leqslant M$), we denote by u_n^s the number of balls with index $j \equiv s \bmod M$ in the nth box at time step $t \equiv \left[\frac{s-1}{M}\right]$, where $[x]$ denotes the largest integer which does not exceed x. In other words, the new *time* variable s is a refinement of the original time, indicating explicitly when balls with index j will move.

We assume that θ_n and u_n^s satisfy the relation

$$\sum_{n=1}^{N} \theta_n - \sum_{j=1}^{M} \sum_{n=1}^{N} u_n^j \geqslant \sum_{n=1}^{N} u_n^k \qquad (k = 1, 2, \ldots, M). \tag{2.1}$$

The first and second terms of the left-hand side of (2.1) represent the number of spaces and the number of balls in the PBBS respectively, hence the left-hand side is nothing but the total number of free spaces of the PBBS. The right-hand side of (2.1) is the number of balls with index k. Thus (2.1) requires the total number of free spaces of the PBBS to be larger than the number of copies of any type of ball in the time evolution process.

Example 2.1. In figure 3, $N = 5$, $M = 2$, $\theta_1 = \theta_2 = \theta_5 = 2$, $\theta_3 = \theta_4 = 1$ and u_n^s are given

$$\begin{array}{lllllll} s = 1 & : & u_{n=1}^{s=1} = 1 & u_2^1 = 1 & u_3^1 = 0 & u_4^1 = 0 & u_5^1 = 0 \\ s = 2 & : & u_1^2 = 0 & u_2^2 = 1 & u_3^2 = 1 & u_4^2 = 0 & u_5^2 = 0. \end{array}$$

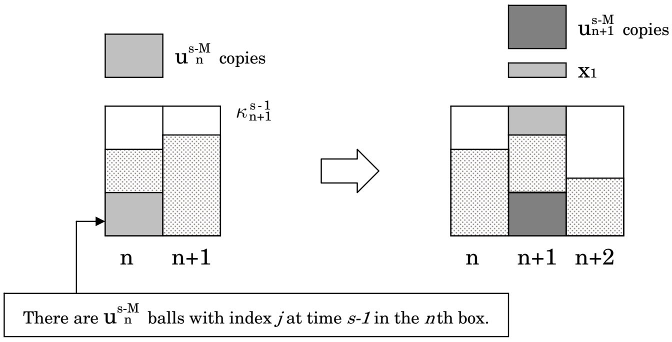

Figure 4. u_n^{s-M}, κ_{n+1}^{s-1} and x_1.

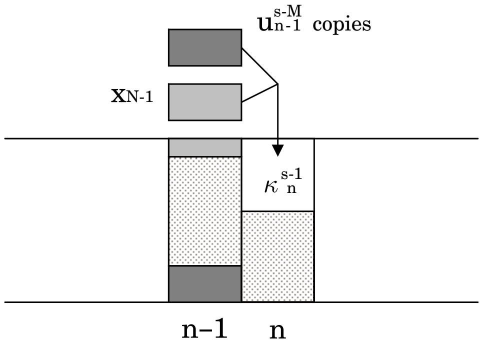

Figure 5. u_n^s.

Let us consider the process at time s, i.e., the movement of the balls with index j at time step t where $s = Mt + j$; we often use s instead of j, i.e. we treat the indices modulo M. If we define κ_n^s, which denotes the number of spaces of the nth box at s, by

$$\kappa_n^s := \theta_n - \left(u_n^s + u_n^{s-1} + \cdots + u_n^{s-M+1}\right) \tag{2.2}$$

condition (2.1) is rewritten as

$$\sum_{n=1}^{N} u_n^{s-M} \leqslant \sum_{n=1}^{N} \kappa_n^{s-1}. \tag{2.3}$$

Since u_n^{s-M} is the number of balls with index s in the nth box at time $s-1$, if we introduce

$$x_1 := \max\left[0, u_n^{s-M} - \kappa_{n+1}^{s-1}\right] \tag{2.4}$$

x_1 will be the number of balls which overflow the $(n+1)$th box, as shown in figure 4. Hence $x_1 + u_{n+1}^{s-M}$ balls will move in the next step. By the same idea, we define

$$\begin{aligned} x_2 &:= \max\left[0, x_1 + u_{n+1}^{s-M} - \kappa_{n+2}^{s-1}\right] \\ x_3 &:= \max\left[0, x_2 + u_{n+2}^{s-M} - \kappa_{n+3}^{s-1}\right] \\ &\vdots \\ x_{N-1} &:= \max\left[0, x_{N-2} + u_{n+N-2}^{s-M} - \kappa_{n+N-1}^{s-1}\right]. \end{aligned} \tag{2.5}$$

Hence, u_n^s, which is the number of balls going into the nth box at time s as shown in figure 5, is given by

$$u_n^s = \min\left[\kappa_n^{s-1}, x_{N-1} + u_{n+N-1}^{s-M}\right]$$
$$= \kappa_n^{s-1} - \max\left[0, \kappa_n^{s-1} - x_{N-1} - u_{n+N-1}^{s-M}\right]. \quad (2.6)$$

Using (2.3), (2.4), (2.5), (2.6) and associativity and distributivity of the operations 'max' and '+', i.e., for $^\forall a, b, c \in \mathbb{R}$,

$$\max[a, \max[b, c]] = \max[a, b, c] \qquad a + \max[b, c] = \max[a + b, a + c]$$

we have the following theorem.

Theorem 2.1. *The time evolution of the PBBS is described by an ultradiscrete equation:*

$$u_n^s - \kappa_n^{s-1} = \alpha - \max[0, \widetilde{\alpha}] \quad (2.7)$$

where

$$\alpha = \max\left[u_{n-1}^{s-M} - \kappa_n^{s-1}, u_{n-1}^{s-M} + u_{n-2}^{s-M} - \kappa_n^{s-1} - \kappa_{n-1}^{s-1}, \ldots, \sum_{j=1}^{N} u_{n-j}^{s-M} - \kappa_{n+1-j}^{s-1}\right] \quad (2.8)$$

$$\widetilde{\alpha} = \max\left[u_{n-1}^{s-M} - \kappa_n^{s-1}, u_{n-1}^{s-M} + u_{n-2}^{s-M} - \kappa_n^{s-1} - \kappa_{n-1}^{s-1}, \ldots, \sum_{j=1}^{N-1} u_{n-j}^{s-M} - \kappa_{n+1-j}^{s-1}\right]. \quad (2.9)$$

3. Reduction of the ndKP equation and its Lax representation

The nonautonomous discrete KP (ndKP) equation [17] is obtained from the generating formula of the KP hierarchy [18, 19]. For $g \in GL_\infty$, we define a tau function $\tau(\boldsymbol{x})$ and a wavefunction $\psi_\lambda(\boldsymbol{x})$ as

$$\tau(\boldsymbol{x}) := \langle \mathrm{vac}|\, \mathrm{e}^{H(\boldsymbol{x})} g |\mathrm{vac}\rangle \quad (3.1)$$

$$\psi_\lambda(\boldsymbol{x}) := \frac{\tau\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right)}{\tau(\boldsymbol{x})}\, \mathrm{e}^{\xi(\boldsymbol{x},\lambda)} \quad (3.2)$$

where

$$\boldsymbol{x} = (x_1, x_2, x_3, \ldots) \qquad \boldsymbol{\epsilon}(1/\lambda) := (1/\lambda, 1/(2\lambda^2), 1/(3\lambda^3), 1/(4\lambda^4), \ldots)$$
$$\xi(\boldsymbol{x}, \lambda) := \sum_{n=1}^{\infty} x_n \lambda^n.$$

To obtain the ndKP equation, we put

$$\boldsymbol{x} = \sum_i^l \boldsymbol{\epsilon}\left(\frac{1}{a(i)}\right) + \sum_j^m \boldsymbol{\epsilon}\left(\frac{1}{b(j)}\right) + \sum_k^n \boldsymbol{\epsilon}\left(\frac{1}{c(k)}\right). \quad (3.3)$$

Here the symbol $\sum_i^l$ denotes the convention

$$\sum_i^l x_i := \begin{cases} \sum_{i=1}^l x_i & l \geqslant 1 \\ 0 & l = 0 \\ -\sum_{i=l+1}^0 x_i & l \leqslant -1. \end{cases}$$

Then the tau function $\tau(l, m, n)$ and the wavefunction $\psi_\lambda(l, m, n)$ for the ndKP equation is given as

$$\tau(l,m,n) := \tau(\boldsymbol{x}) \tag{3.4}$$

$$\psi_\lambda(l,m,n) := \psi_\lambda(\boldsymbol{x}) \tag{3.5}$$

where $\boldsymbol{x}$ is given in (3.3). Then, from the generating formula of the KP hierarchy, we have the Lax representation

$$\begin{cases} \psi_{lm} = \frac{1}{b_m - a_l}\frac{\tau_l\tau_m}{\tau\tau_{lm}}[b_m\psi_l - a_l\psi_m] \\ \psi_{mn} = \frac{1}{c_n - b_m}\frac{\tau_m\tau_n}{\tau\tau_{mn}}[c_n\psi_m - b_m\psi_n] \\ \psi_{nl} = \frac{1}{a_l - c_n}\frac{\tau_n\tau_l}{\tau\tau_{nl}}[a_l\psi_n - c_n\psi_l] \end{cases} \tag{3.6}$$

and the compatibility condition of (3.6) gives the ndKP equation

$$(b_m - c_n)\tau_l\tau_{mn} + (c_n - a_l)\tau_m\tau_{nl} + (a_l - b_m)\tau_n\tau_{lm} = 0. \tag{3.7}$$

Here we use the abbreviations $\tau_l \equiv \tau(l+1,m,n)$, $\tau_{lm} \equiv \tau(l+1,m+1,n)$, $\psi_l \equiv \psi_\lambda(l+1,m,n)$, $\tau_{l'} \equiv \tau(l-1,m,n)$, $\tau_{l'm'} \equiv \tau(l-1,m-1,n)$, $a_l \equiv a(l+1)$ etc.

In order to relate the ndKP equation to the PBBS, we take $a(l) = 0$, $b(m) = 1$, $c(n) = 1 + \delta_n$ and impose the following constraint on $\tau(l,m,n)$:

$$\tau(l,m,n) = \tau(l-M, m-1, n). \tag{3.8}$$

If we set $l+1 = s$, $m = 0$ and $\sigma_n^s := \tau(s-1, 0, n)$, (3.7) turns into

$$-\delta_{n+1}\sigma_n^{s+1}\sigma_{n+1}^{s-M} + (1+\delta_{n+1})\sigma_n^{s-M}\sigma_{n+1}^{s+1} - \sigma_{n+1}^{s}\sigma_n^{s-M+1} = 0. \tag{3.9}$$

To impose the above conditions on (3.6), we have to rescale the wavefunction due to the condition $a(l) = 0$:

$$\psi'_\lambda := \lim_{a\to 0}\frac{1}{(-a)^l}\psi_\lambda.$$

Then, from (3.2) and (3.8), $\psi'_\lambda(l,m,n)$ satisfies

$$\psi'_\lambda(l,m,n) = \frac{1}{\lambda^M(1-\lambda)}\psi'_\lambda(l-M, m-1, n) \tag{3.10}$$

and (3.6) turns into

$$\begin{cases} \psi'_{lm} = \frac{\tau_l\tau_m}{\tau\tau_{lm}}[\psi'_l + \psi'_m] \\ \psi'_{mn} = \frac{1}{\delta_{n+1}}\frac{\tau_m\tau_n}{\tau\tau_{mn}}[(1+\delta_{n+1})\psi'_m - \psi'_n] \\ \psi'_{nl} = \frac{1}{1+\delta_{n+1}}\frac{\tau_n\tau_l}{\tau\tau_{nl}}[\psi'_n - (1+\delta_{n+1})\psi'_l]. \end{cases} \tag{3.11}$$

If we define $\varphi_n^s := \psi'(s-1, 0, n)$, we have

$$\begin{cases} \varphi_n^{s+1} = A_n^s\left[\Lambda\varphi_n^{s+M+1} + \varphi_n^s\right] \\ \varphi_{n+1}^s = B_n^s\left[\left(1 + \frac{1}{\delta_{n+1}}\right)\varphi_n^s + \frac{1}{\delta_{n+1}}\Lambda\varphi_{n+1}^{s+M}\right] \\ \varphi_{n+1}^{s+1} = U_n^s\left[\frac{1}{1+\delta_{n+1}}\varphi_{n+1}^s + \varphi_n^{s+1}\right] \end{cases} \tag{3.12}$$

where

$$\Lambda := \lambda^M(1-\lambda) \qquad A_n^s = \frac{\sigma_n^{s+M+1}\sigma_n^s}{\sigma_n^{s+M}\sigma_n^{s+1}} \qquad B_n^s = \frac{\sigma_n^s\sigma_{n+1}^{s+M}}{\sigma_n^{s+M}\sigma_{n+1}^s} \qquad U_n^s = \frac{\sigma_{n+1}^s\sigma_n^{s+1}}{\sigma_n^s\sigma_{n+1}^{s+1}}.$$

The compatibility condition of (3.12) gives

$$\frac{1}{A_{n+1}^{s-1}} - \frac{1+\delta_{n+1}}{U_n^{s-1}} = -\frac{\delta_{n+1}}{B_n^s} \tag{3.13}$$

which is, of course, equivalent to (3.9).

4. From the ndKP equation to the PBBS

We will show that the ultradiscrete limit of (3.13) coincides with the PBBS. First we express A_n^s and B_n^s in terms of U_n^s as

$$\begin{aligned} B_n^s &= \frac{\sigma_n^s \sigma_{n+1}^{s+M}}{\sigma_{n+1}^s \sigma_n^{s+M}} = \frac{\sigma_n^s \sigma_{n+1}^{s+1}}{\sigma_{n+1}^s \sigma_n^{s+1}} \cdots \frac{\sigma_n^{s+M-1} \sigma_{n+1}^{s+M}}{\sigma_{n+1}^{s+M-1} \sigma_n^{s+M}} \\ &= \left(U_n^s U_n^{s+1} \cdots U_n^{s+M-1}\right)^{-1} \\ &= \left(\prod_{j=0}^{M-1} U_n^{s+M-j-1}\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} A_{n+1}^{s-1} &= \left(\frac{1+\delta_{n+1}}{U_n^{s-1}} - \frac{\delta_{n+1}}{B_n^s}\right)^{-1} \\ &= \left(\frac{1+\delta_{n+1}}{U_n^{s-1}} - \delta_{n+1} \prod_{j=0}^{M-1} U_n^{s+M-j-1}\right)^{-1} \\ &= U_n^{s-1} \left(1+\delta_{n+1} - \delta_{n+1} \prod_{j=0}^{M} U_n^{s+M-j-1}\right)^{-1}. \end{aligned} \tag{4.1}$$

In analogy with (2.2), we define a new variable K_n^s by

$$\frac{1}{K_n^s} := \delta_{n+1} \cdot \prod_{j=1}^{M} U_n^{s-j+1}.$$

Then (4.1) turns into

$$A_{n+1}^{s-1} = U_n^{s-1} \left(1+\delta_{n+1} - \frac{U_n^{s+M-1}}{K_n^{s+M-2}}\right)^{-1}.$$

Since

$$\begin{aligned} \frac{A_{n+1}^{s-1}}{A_n^{s-1}} &= \frac{\sigma_{n+1}^{s+M} \sigma_{n+1}^{s-1}}{\sigma_{n+1}^{s+M-1} \sigma_{n+1}^{s}} \cdot \frac{\sigma_n^{s+M-1} \sigma_n^{s}}{\sigma_n^{s+M} \sigma_n^{s-1}} \\ &= \frac{\sigma_{n+1}^{s-1} \sigma_n^{s}}{\sigma_n^{s-1} \sigma_{n+1}^{s}} \cdot \frac{\sigma_n^{s+M-1} \sigma_{n+1}^{s+M}}{\sigma_{n+1}^{s+M-1} \sigma_n^{s+M}} \\ &= \frac{U_n^{s-1}}{U_n^{s+M-1}} \end{aligned}$$

we obtain

$$\frac{U_n^s}{U_n^{s+M}} = \frac{U_n^s}{U_{n-1}^s} \left(1+\delta_n - \frac{U_{n-1}^{s+M}}{K_{n-1}^{s+M-1}}\right) \left(1+\delta_{n+1} - \frac{U_n^{s+M}}{K_n^{s+M-1}}\right)^{-1}$$

which is equivalent to

$$\frac{U_n^{s+M}}{K_n^{s+M-1}} = \frac{1+\delta_{n+1}}{1 + \frac{K_n^{s+M-1}}{U_{n-1}^s} \left(1+\delta_n - \frac{U_{n-1}^{s+M}}{K_{n-1}^{s+M-1}}\right)}.$$

If we set $\widetilde{U}_n^s := U_n^s \big/ (1 + \delta_{n+1})$, we have

$$\frac{\widetilde{U}_n^{s+M}}{K_n^{s+M-1}} = \frac{1}{1 + \frac{K_n^{s+M-1}}{\widetilde{U}_{n-1}^s}\left(1 - \frac{\widetilde{U}_{n-1}^{s+M}}{K_{n-1}^{s+M-1}}\right)}. \tag{4.2}$$

Now we impose a periodic condition on U_n^s:

$$U_n^s = U_{n+N}^s. \tag{4.3}$$

Solving (4.2) with respect to $\frac{\widetilde{U}_n^{s+M}}{K_n^{s+M-1}}$, we obtain

$$\frac{\widetilde{U}_n^{s+M}}{K_n^{s+M-1}} = \frac{\chi_n^s}{1 + \widetilde{\chi}_n^s} \tag{4.4}$$

where

$$\chi_n^s = \frac{\widetilde{U}_{n-1}^s}{K_n^{s+M-1}} + \frac{\widetilde{U}_{n-1}^s \widetilde{U}_{n-2}^s}{K_n^{s+M-1} K_{n-1}^{s+M-1}} + \cdots + \frac{\widetilde{U}_{n-1}^s \cdots \widetilde{U}_{n-N}^s}{K_n^{s+M-1} \cdots K_{n-N+1}^{s+M-1}} \tag{4.5}$$

$$\widetilde{\chi}_n^s = \frac{\widetilde{U}_{n-1}^s}{K_n^{s+M-1}} + \frac{\widetilde{U}_{n-1}^s \widetilde{U}_{n-2}^s}{K_n^{s+M-1} K_{n-1}^{s+M-1}} + \cdots + \frac{\widetilde{U}_{n-1}^s \cdots \widetilde{U}_{n-N+1}^s}{K_n^{s+M-1} \cdots K_{n-N+2}^{s+M-1}}. \tag{4.6}$$

To take the ultradiscrete limit, we put $U_n^s = \mathrm{e}^{u_n^s/\epsilon}$, $K_n^s = \mathrm{e}^{\kappa_n^s/\epsilon}$, $1/\delta_{n+1} = \mathrm{e}^{\theta_n/\epsilon}$. Since

$$\widetilde{U}_n^s = \frac{U_n^s}{1 + \delta_{n+1}} = \mathrm{e}^{u_n^s/\epsilon} \cdot (1 + \mathrm{e}^{-\theta_n/\epsilon})^{-1}$$

we have that the ultradiscrete of $\widetilde{U}_n^s$ is nothing but the variable u_n^s

$$\lim_{\epsilon \to 0^+} \epsilon \log \widetilde{U}_n^s = u_n^s - \max(0, -\theta_n) = u_n^s.$$

Therefore one can easily see that the ultradiscrete limit of (4.4) turns into (2.7).

In conclusion, we have proved

Theorem 4.1. *The ultradiscrete limit of the constrained ndKP equation (3.9) (or (3.13)) with the periodic boundary condition (4.3) coincides with the time evolution equation of the PBBS (2.7).*

5. Conserved quantities of the PBBS

In this section, we consider the conserved quantities of the ndKP equation (3.13) with respect to the *time* variable s. Taking ultradiscrete limits of them, we will obtain the conserved quantities of the PBBS.

We imposed the periodic boundary condition (4.3) for U_n^s. Accordingly we assume a boundary condition for the wavefunction φ_n^s:

$$\varphi_n^s = \eta \varphi_{n+N}^s. \tag{5.1}$$

Here η is a parameter independent of Λ. Equations (3.12) and (5.1) yield

$$\begin{cases} \widetilde{L}(s)\varphi^s = \Lambda \varphi^{s+M} \\ \widetilde{M}(s+1)\varphi^{s+1} = \varphi^s \end{cases}$$

where

$$\widetilde{L}(s-M) := \begin{bmatrix} -\frac{\delta_1}{B_N^{s-M}} & & & & (1+\delta_1)\eta \\ 1+\delta_2 & -\frac{\delta_2}{B_1^{s-M}} & & & \\ & 1+\delta_3 & -\frac{\delta_3}{B_2^{s-M}} & & \\ & & \ddots & \ddots & \\ & & & 1+\delta_N & -\frac{\delta_N}{B_{N-1}^{s-M}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{K_N^{s-1}} & & & & (1+\delta_1)\eta \\ 1+\delta_2 & -\frac{1}{K_1^{s-1}} & & & \\ & 1+\delta_3 & -\frac{1}{K_2^{s-1}} & & \\ & & \ddots & \ddots & \\ & & & 1+\delta_N & -\frac{1}{K_{N-1}^{s-1}} \end{bmatrix} \tag{5.2}$$

$$\widetilde{M}(s) := \begin{bmatrix} \frac{1+\delta_1}{U_N^{s-1}} & & & & -(1+\delta_1)\eta \\ -(1+\delta_2) & \frac{1+\delta_2}{U_1^{s-1}} & & & \\ & -(1+\delta_3) & \frac{1+\delta_3}{U_2^{s-1}} & & \\ & & \ddots & \ddots & \\ & & & -(1+\delta_N) & \frac{1+\delta_N}{U_{N-1}^{s-1}} \end{bmatrix} \tag{5.3}$$

and $\varphi^s := {}^t(\varphi_1^s, \varphi_2^s, \ldots, \varphi_N^s)$. Hence, by putting

$$\widehat{L}(M;s) = \widetilde{L}(s-M)\widetilde{M}(s-M+1)\widetilde{M}(s-M+2)\cdots\widetilde{M}(s) \tag{5.4}$$

we obtain

$$\begin{cases} \widehat{L}(M;s)\varphi^s = \Lambda\varphi^s \\ \widetilde{M}(s+1)\varphi^{s+1} = \varphi^s. \end{cases} \tag{5.5}$$

From (5.5), for arbitrary $\Lambda \in \mathbb{C}$ and the corresponding wavefunction $\varphi^s \equiv \varphi^s_\Lambda$, we have

$$\widehat{L}(M;s+1)\widetilde{M}^{-1}(s+1)\varphi^s = \widetilde{M}^{-1}(s+1)\widehat{L}(M;s)\varphi^s$$

which yields

$$\widehat{L}(M;s+1)\widetilde{M}^{-1}(s+1) = \widetilde{M}^{-1}(s+1)\widehat{L}(M;s)$$

or equivalently

$$\widehat{L}(M;s+1) = \widetilde{M}^{-1}(s+1)\widehat{L}(M;s)\widetilde{M}(s+1). \tag{5.6}$$

Hence

$$\begin{aligned} \det(\lambda I + \widehat{L}(M;s+1)) &= \det(\lambda I + \widetilde{M}^{-1}(s+1)\widehat{L}(M;s)\widetilde{M}(s+1)) \\ &= \det(\lambda I + \widehat{L}(M;s)) \end{aligned}$$

where I is the $N \times N$ unit matrix. Therefore, when we expand the determinant with respect to λ

$$\det(\lambda I + \widehat{L}(M;s)) = \lambda^N + e_{N-1}\lambda^{N-1} + e_{N-2}\lambda^{N-2} + \cdots + e_1\lambda + e_0 \tag{5.7}$$

the coefficients $e_k = e_k\big(\{U_n^s\}\big)$ $(k = 0, 1, \ldots, N-1)$ are conserved in time s. Note that e_k is equal to the $(N-k)$th fundamental symmetric function of the eigenvalues of the matrix $\widehat{L}(M; s)$. In the ultradiscrete limit, e_k will be converted into a conserved quantity of the PBBS.

Remark 5.1. As before, we introduce $\widetilde{U}_n^s = U_n^s \big/ (1+\delta_{n+1})$. For $N \geqslant M+2$, the (n, m) element of $\widehat{L}(M; s)$ is

(i) if $m = n + N - M - 1 \pmod N$,

$$\begin{cases} (-1)^M \eta \cdot \prod_{i=1}^{M+1} (1+\delta_{n+N-M+i-1}) & (1 \leqslant n \leqslant M+1) \\ (-1)^M \prod_{i=1}^{M+1} (1+\delta_{n+N-M+i-1}) & (\text{otherwise}) \end{cases}$$

(ii) if $m = n + N - M \pmod N$,

$$\begin{cases} (-1)^{M+1} \eta \cdot \left(\frac{1}{K_{n-1}^{s-1}} + \sum_{j=1}^{M} \frac{1}{\widetilde{U}_{n-M+j-2}^{s-j}} \right) \cdot \prod_{i=1}^{M} (1+\delta_{n+N-M+i}) & (1 \leqslant n \leqslant M) \\ (-1)^{M+1} \left(\frac{1}{K_{n-1}^{s-1}} + \sum_{j=1}^{M} \frac{1}{\widetilde{U}_{n-M+j-2}^{s-j}} \right) \cdot \prod_{i=1}^{M} (1+\delta_{n+N-M+i}) & (\text{otherwise}) \end{cases}$$

(iii) if $m = n + N - M + k - 1 \pmod N$ $(k = 2, 3, \ldots, M)$,

$$\begin{cases} (-1)^{M+k} \eta \cdot \prod_{i=1}^{M-k+1} (1+\delta_{n+N-M+k+i-1}) & \\ \quad \times \left(\frac{1}{K_{n-1}^{s-1}} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{k-1} \leqslant M} \frac{1}{\widetilde{U}_{n-M+j_1+k-3}^{s-j_1} \widetilde{U}_{n-M+j_2+k-4}^{s-j_2} \cdots \widetilde{U}_{n-M+j_{k-1}-1}^{s-j_{k-1}}} \right. & \\ \quad \left. + \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant M} \frac{1}{\widetilde{U}_{n-M+j_1+k-3}^{s-j_1} \widetilde{U}_{n-M+j_2+k-4}^{s-j_2} \cdots \widetilde{U}_{n-M+j_k-2}^{s-j_k}} \right) & (1 \leqslant n \leqslant M-k+1) \\ (-1)^{M+k} \prod_{i=1}^{M-k+1} (1+\delta_{n+N-M+k+i-1}) & \\ \quad \times \left(\frac{1}{K_{n-1}^{s-1}} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{k-1} \leqslant M} \frac{1}{\widetilde{U}_{n-M+j_1+k-3}^{s-j_1} \widetilde{U}_{n-M+j_2+k-4}^{s-j_2} \cdots \widetilde{U}_{n-M+j_{k-1}-1}^{s-j_{k-1}}} \right. & \\ \quad \left. + \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant M} \frac{1}{\widetilde{U}_{n-M+j_1+k-3}^{s-j_1} \widetilde{U}_{n-M+j_2+k-4}^{s-j_2} \cdots \widetilde{U}_{n-M+j_k-2}^{s-j_k}} \right) & (\text{otherwise}) \end{cases}$$

(iv) if $m = n$, $-\frac{1}{\Theta_{n-1}}$
where

$$\Theta_n := K_n^{s-1} \cdot \prod_{j=1}^{M} \widetilde{U}_n^{s-M+j-1} = \frac{1}{\delta_{n+1}} \left(\frac{1}{1+\delta_{n+1}} \right)^M$$

(v) othewise, 0.

To see this, it is convenient to use the formula

$$\begin{aligned} \widehat{L}(M; s) &= (-\widetilde{D}_K(s-M) + \widetilde{\Upsilon})(\widetilde{D}_{\widetilde{U}}(s-M+1) - \widetilde{\Upsilon}) \\ &\quad \times (\widetilde{D}_{\widetilde{U}}(s-M+2) - \widetilde{\Upsilon}) \cdots (\widetilde{D}_{\widetilde{U}}(s) - \widetilde{\Upsilon}) \\ &= -\widetilde{D}_K(s-M)\widetilde{D}_{\widetilde{U}}(s-M+1) \cdots \widetilde{D}_{\widetilde{U}}(s) \\ &\quad + (\widetilde{D}_K(s-M)\widetilde{D}_{\widetilde{U}}(s-M+1) \cdots \widetilde{D}_{\widetilde{U}}(s-1)\widetilde{\Upsilon} \\ &\quad + \widetilde{D}_K(s-M)\widetilde{D}_{\widetilde{U}}(s-M+1) \cdots \widetilde{D}_{\widetilde{U}}(s-2)\widetilde{\Upsilon}\widetilde{D}_{\widetilde{U}}(s) \\ &\quad + \cdots + \widetilde{\Upsilon}\widetilde{D}_{\widetilde{U}}(s-M+1) \cdots \widetilde{D}_{\widetilde{U}}(s)) + \cdots + (-1)^M \widetilde{\Upsilon}^{M+1} \end{aligned}$$

where

$$\widetilde{D}_K(s-M) := \begin{bmatrix} \frac{1}{K_N^{s-1}} & & & & \\ & \frac{1}{K_1^{s-1}} & & & \\ & & \frac{1}{K_2^{s-1}} & & \\ & & & \ddots & \\ & & & & \frac{1}{K_{N-1}^{s-1}} \end{bmatrix}$$

$$\widetilde{D}_{\widetilde{U}}(s) := \begin{bmatrix} \frac{1}{\widetilde{U}_N^{s-1}} & & & & \\ & \frac{1}{\widetilde{U}_1^{s-1}} & & & \\ & & \frac{1}{\widetilde{U}_2^{s-1}} & & \\ & & & \ddots & \\ & & & & \frac{1}{\widetilde{U}_{N-1}^{s-1}} \end{bmatrix}$$

$$\widetilde{\Upsilon} := \begin{bmatrix} & & & & (1+\delta_1)\cdot\eta \\ 1+\delta_2 & & & & \\ & 1+\delta_3 & & & \\ & & \ddots & & \\ & & & 1+\delta_N & \end{bmatrix}.$$

In this paper, we will mainly treat the case $M = 1$ and we do not consider the case $N < M + 2$ which is almost trivial.

6. Conserved quantities of the ndKP equation in the case $M = 1$

Hereafter, we restrict ourselves to the case $M = 1$. In this case the conserved quantity e_k $(k = 1, 2, \ldots, N)$ is the coefficient of λ^k in the expansion of $\det(\lambda I + \widehat{L}(1; s))$. First we have the following lemma.

Lemma 6.1. *Let*

$$\widetilde{\Upsilon}_0 := \begin{bmatrix} & & & & \prod_{j=1}^N(1+\delta_j)\cdot\eta \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}$$

and

$$\widehat{L}_0(M; s) := (-\widetilde{D}_K(s-M) + \widetilde{\Upsilon}_0)(\widetilde{D}_{\widetilde{U}}(s-M+1) - \widetilde{\Upsilon}_0) \times (\widetilde{D}_{\widetilde{U}}(s-M+2) - \widetilde{\Upsilon}_0)\cdots(\widetilde{D}_{\widetilde{U}}(s) - \widetilde{\Upsilon}_0).$$

Then, it holds that

$$\det(\lambda I + \widehat{L}(M; s)) = \det(\lambda I + \widehat{L}_0(M; s)). \tag{6.1}$$

This lemma follows immediately from the identity

$$\widetilde{\Upsilon}_0 = (\widetilde{D}_\Delta)^{-1}\widetilde{\Upsilon}\widetilde{D}_\Delta$$

where

$$\widetilde{D}_\Delta := \begin{bmatrix} 1 & & & & \\ & 1+\delta_2 & & & \\ & & (1+\delta_2)(1+\delta_3) & & \\ & & & \ddots & \\ & & & & \prod_{j=2}^{N}(1+\delta_j) \end{bmatrix}.$$

From lemma 6.1 we find

$$\det(\lambda I + \widehat{L}(1;s)) = \det(\lambda I + \widehat{L}_0(1;s)) = \begin{vmatrix} a_1 & & & & -\eta\cdot\Delta & \eta\cdot b_N\Delta \\ b_1 & a_2 & & & & -\eta\cdot\Delta \\ -1 & b_2 & \ddots & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{N-1} & \\ & & & -1 & b_{N-1} & a_N \end{vmatrix} \tag{6.2}$$

where

$$a_n := \lambda - \frac{1}{\Theta_{n-1}} \qquad b_n := \frac{1}{K_n^{s-1}} + \frac{1}{\widetilde{U}_{n-1}^{s-1}} \qquad \Delta := \prod_{j=1}^{N}(1+\delta_j). \tag{6.3}$$

Expanding the determinant (6.2) yields

$$\det(\lambda I + \widehat{L}(1;s)) = \prod_{n=1}^{N}\left(\lambda - \frac{1}{\Theta_n}\right) + (-1)^{N+1}\eta\cdot B_N\Delta + (-1)^N\eta^2\Delta^2 \tag{6.4}$$

where

$$B_N := B(1,N) + \left(\lambda - \frac{1}{\Theta_N}\right)B(2,N-1) \tag{6.5}$$

and

$$B(m,n) := \begin{vmatrix} b_m & a_{m+1} & & & & \\ -1 & b_{m+1} & a_{m+2} & & & \\ & -1 & b_{m+2} & \ddots & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & \ddots & a_{n-1} & \\ & & & & \ddots & b_{n-1} & a_n \\ & & & & & -1 & b_n \end{vmatrix} \qquad (m < n).$$

Here we have used a relation

$$B(m,n) = \left(\frac{1}{K_n^{s-1}} + \frac{1}{\widetilde{U}_{n-1}^{s-1}}\right)B(m,n-1) + \left(\lambda - \frac{1}{\Theta_{n-1}}\right)B(m,n-2). \tag{6.6}$$

Using (6.6) recursively, we find that

$$B(1,N) = \sum_{(i_1,\ldots,i_p,j_1,\ldots,j_q)} \left(\frac{1}{K_{i_1}} + \frac{1}{\widetilde{U}_{i_1-1}}\right)\cdots\left(\frac{1}{K_{i_p}} + \frac{1}{\widetilde{U}_{i_p-1}}\right)\left(\lambda - \frac{1}{\Theta_{j_1}}\right)\cdots\left(\lambda - \frac{1}{\Theta_{j_q}}\right) \tag{6.7}$$

where the summation is over all $p+q$ tuples $(i_1,\ldots,i_p,j_1,\ldots,j_q)$ of integers which satisfy the following condition:

$$\begin{cases} p+2q=N \qquad p\geqslant 0 \qquad q\geqslant 0 \\ 1\leqslant i_1<i_2<\cdots<i_p\leqslant N \\ 1\leqslant j_1<j_2<\cdots<j_q\leqslant N-1 \\ i_1,i_2,\ldots,i_p, \\ \quad j_1,j_1+1,j_2,j_2+1,\ldots,j_q,j_q+1 \text{ are distinct integers.} \end{cases} \tag{6.8}$$

Hereafter we put $K_n\equiv K_n^{s-1}$ and $\widetilde{U}_n\equiv\widetilde{U}_n^{s-1}$ for simplicity.

In a similar manner, we have an expression for $B(2,N-1)$ and thus we obtain

$$B_N=\sum_{(i_1,\ldots,i_p,j_1,\ldots,j_q)}\left(\frac{1}{K_{i_1}}+\frac{1}{\widetilde{U}_{i_1-1}}\right)\cdots\left(\frac{1}{K_{i_p}}+\frac{1}{\widetilde{U}_{i_p-1}}\right)\left(\lambda-\frac{1}{\Theta_{j_1}}\right)\cdots\left(\lambda-\frac{1}{\Theta_{j_q}}\right) \tag{6.9}$$

where the summation is over all $p+q$ tuples $(i_1,\ldots,i_p,j_1,\ldots,j_q)$ of integers satisfying

$$\begin{cases} p+2q=N \qquad p\geqslant 0 \qquad q\geqslant 0 \\ 1\leqslant i_1<i_2<\cdots<i_p\leqslant N \\ 1\leqslant j_1<j_2<\cdots<j_q\leqslant N \\ i_1,i_2,\ldots,i_p, \\ \quad j_1,j_1+1,j_2,j_2+1,\ldots,j_q,j_q+1 \text{ are distinct modulo } N. \end{cases} \tag{6.10}$$

The main theorem of this section is

Theorem 6.1.

$$B_N=\begin{cases} \sum_{k=0}^{\frac{N}{2}-1} f_k\lambda^k+2\lambda^{\frac{N}{2}} & (N: \mathit{even}) \\ \sum_{k=0}^{\frac{N-1}{2}} f_k\lambda^k & (N: \mathit{odd}) \end{cases} \tag{6.11}$$

where

$$f_k=\sum_{(x_1,x_2,\ldots,x_N)\in\mathcal{K}_{N;N-2k}} x_1x_2\cdots x_N \tag{6.12}$$

and, for $p\geqslant 1$,

$$\mathcal{K}_{N;p}:=\left\{(x_1,x_2,\ldots,x_N)\;\middle|\; \begin{array}{l} x_n\in\{1/K_n,1/\widetilde{U}_n,1\}\textit{ for each } n\textit{, and} \\ \sharp\{n\mid 1\leqslant n\leqslant N, x_n\neq 1\}=p. \\ \textit{Let } x_{i_1},x_{i_2},\ldots,x_{i_p}\ (i_1<i_2<\cdots<i_p) \\ \textit{be the non-1 elements; these then satisfy} \\ \textit{the following conditions: for each } m<p \\ \quad \textit{(i) If } i_{m+1}-i_m \textit{ is odd then} \\ \quad (x_{i_m},x_{i_{m+1}})=(1/K_{i_m},1/K_{i_{m+1}}) \\ \qquad\qquad \textit{or } (1/\widetilde{U}_{i_m},1/\widetilde{U}_{i_{m+1}}). \\ \quad \textit{(ii) Otherwise} \\ \quad (x_{i_m},x_{i_{m+1}})=(1/K_{i_m},1/\widetilde{U}_{i_{m+1}}) \\ \qquad\qquad \textit{or } (1/\widetilde{U}_{i_m},1/K_{i_{m+1}}). \end{array}\right\}.$$

The proof goes as follows.

For simplicity, put $\alpha_n=1/K_n$, $\beta_n=1/\widetilde{U}_{n-1}$; thus

$$B(1,N)=\sum_{(i_1,\ldots,i_p,j_1,\ldots,j_q)}\left(\lambda-\alpha_{j_1}\beta_{j_1+1}\right)\cdots\left(\lambda-\alpha_{j_q}\beta_{j_q+1}\right)\left(\alpha_{i_1}+\beta_{i_1}\right)\cdots\left(\alpha_{i_p}+\beta_{i_p}\right). \tag{6.13}$$

We again need to introduce some more notation. For $p \geqslant 1$,

$$\mathcal{B}_{N;p} := \left\{ (x_1, x_2, \ldots, x_N) \;\middle|\; \begin{array}{l} x_n \in \{\alpha_n, \beta_n, 1\} \text{ for each } n, \text{ and} \\ \sharp\{n \mid 1 \leqslant n \leqslant N, x_n \neq 1\} = p. \\ \text{Let } x_{i_1}, x_{i_2}, \ldots, x_{i_p} \ (i_1 < i_2 < \cdots < i_p) \\ \text{be the non-1 elements; then, for each } m < p \\ i_{m+1} - i_m \text{ is odd, and if } i_{m+1} - i_m = 1 \\ \text{then } \left(x_{i_m}, x_{i_{m+1}}\right) \neq \left(\alpha_{i_m}, \beta_{i_{m+1}}\right). \end{array} \right\}.$$

Define $\iota_1, \iota_p : \mathcal{B}_{N;p} \to \{1, 2, \ldots, N\}$ by $\iota_1(x_1, x_2, \ldots, x_N) = i_1$ and $\iota_p(x_1, x_2, \ldots, x_N) = i_p$ when $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ $(i_1 < i_2 < \cdots < i_p)$ are the non-1 elements in $(x_1, x_2, \ldots, x_N)$. For $p \geqslant 1$,

$$\mathcal{O}_{N;p} := \{\mathbf{x} \in \mathcal{B}_{N;p} \mid \iota_1(\mathbf{x}) \text{ is an odd integer}\} \tag{6.14}$$

$$\mathcal{E}_{N;p} := \{\mathbf{x} \in \mathcal{B}_{N;p} \mid \iota_1(\mathbf{x}) \text{ is an even integer}\} \tag{6.15}$$

$$\mathcal{P}_{N;p} := \left\{ \mathbf{x} \in \mathcal{B}_{N;p} \;\middle|\; \begin{array}{l} N + \iota_1(\mathbf{x}) - \iota_p(\mathbf{x}) \text{ is an odd integer} \\ \text{and if } N + \iota_1(\mathbf{x}) - \iota_p(\mathbf{x}) = 1 \\ \left(x_{\iota_p(\mathbf{x})}, x_{\iota_1(\mathbf{x})}\right) \neq \left(\alpha_{\iota_p(\mathbf{x})}, \beta_{\iota_1(\mathbf{x})}\right) \end{array} \right\}. \tag{6.16}$$

For $\mathcal{B}' \subset \mathcal{B}_{N;p}$, define $\xi(\mathcal{B}')$ by

$$\xi(\mathcal{B}') := \sum_{(x_1, x_2, \ldots, x_N) \in \mathcal{B}'} x_1 x_2 \cdots x_N. \tag{6.17}$$

Lemma 6.2.

$$B(1, N) = \begin{cases} \sum_{q=0}^{\frac{N}{2}-1} \xi(\mathcal{O}_{N;N-2q}) \cdot \lambda^q + \lambda^{\frac{N}{2}} & (N : \mathit{even}) \\ \sum_{q=0}^{\frac{N-1}{2}} \xi(\mathcal{O}_{N;N-2q}) \cdot \lambda^q & (N : \mathit{odd}). \end{cases} \tag{6.18}$$

Proof. We prove (6.18) by induction on N.

(i) $N = 2$: By (6.13)

$$\begin{aligned} B(1, 2) &= (\alpha_1 + \beta_1)(\alpha_2 + \beta_2) + (\lambda - \alpha_1 \beta_2) \\ &= \lambda + (\alpha_1 \alpha_2 + \beta_1 \alpha_2 + \beta_1 \beta_2). \end{aligned}$$

On the other hand,

$$\xi(\mathcal{O}_{2;2}) = \alpha_1 \alpha_2 + \beta_1 \alpha_2 + \beta_1 \beta_2.$$

Hence (6.18) is true for $N = 2$.

(ii) $N = 3$: by (6.13)

$$\begin{aligned} B(1, 3) &= (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_3) + (\lambda - \alpha_1 \beta_2)(\alpha_3 + \beta_3) + (\lambda - \alpha_2 \beta_3)(\alpha_1 + \beta_1) \\ &= \{(\alpha_1 + \beta_1) + (\alpha_3 + \beta_3)\}\lambda + (\alpha_1 \alpha_2 \alpha_3 + \beta_1 \alpha_2 \alpha_3 + \beta_1 \beta_2 \alpha_3 + \beta_1 \beta_2 \beta_3). \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi(\mathcal{O}_{3;1}) &= (\alpha_1 + \beta_1) + (\alpha_3 + \beta_3) \\ \xi(\mathcal{O}_{3;3}) &= \alpha_1 \alpha_2 \alpha_3 + \beta_1 \alpha_2 \alpha_3 + \beta_1 \beta_2 \alpha_3 + \beta_1 \beta_2 \beta_3. \end{aligned}$$

Hence (6.18) is true for $N = 3$.

(iii) Suppose that (6.18) holds up to $N-1$ ($N \geqslant 4$). By (6.6) and the induction hypothesis, if N is even then

$$\begin{aligned} B(1,N) = {} & (\alpha_N+\beta_N)\xi(\mathcal{O}_{N-1;N-1}) - \alpha_{N-1}\beta_N\cdot\xi(\mathcal{O}_{N-2;N-2}) \\ & + \sum_{q=1}^{\frac{N}{2}-2}\{(\alpha_N+\beta_N)\xi(\mathcal{O}_{N-1;N-2q-1}) + \xi(\mathcal{O}_{N-2;N-2q}) \\ & - \alpha_{N-1}\beta_N\cdot\xi(\mathcal{O}_{N-2;N-2q-2})\}\lambda^q + \{(\alpha_N+\beta_N)\xi(\mathcal{O}_{N-1;1}) \\ & + \xi(\mathcal{O}_{N-2;2}) - \alpha_{N-1}\beta_N\}\lambda^{\frac{N}{2}-1} + \lambda^{\frac{N}{2}}. \end{aligned} \tag{6.19}$$

Since $\mathcal{O}_{N;N-2q}$ for $1 \leqslant q \leqslant \frac{N}{2}-2$ can be decomposed

$$\begin{aligned} \mathcal{O}_{N;N-2q} = {} & \{(x_1,x_2,\ldots,x_{N-2},1,1) \mid (x_1,x_2,\ldots,x_{N-2}) \in \mathcal{O}_{N-2;N-2q}\} \\ & \sqcup \{(x_1,x_2,\ldots,x_{N-1},\alpha_N) \mid (x_1,x_2,\ldots,x_{N-1}) \in \mathcal{O}_{N-1;N-2q-1}\} \\ & \sqcup (\{(x_1,x_2,\ldots,x_{N-1},\beta_N) \mid (x_1,x_2,\ldots,x_{N-1}) \in \mathcal{O}_{N-1;N-2q-1}\} \\ & \setminus \{(x_1,x_2,\ldots,\alpha_{N-1},\beta_N) \mid (x_1,x_2,\ldots,x_{N-2}) \in \mathcal{O}_{N-2;N-2q-2}\}) \end{aligned}$$

we have

$$\begin{aligned} \xi(\mathcal{O}_{N;N-2q}) = {} & \xi(\mathcal{O}_{N-2;N-2q}) + \xi(\mathcal{O}_{N-1;N-2q-1})\cdot\alpha_N \\ & + \xi(\mathcal{O}_{N-1;N-2q-1})\cdot\beta_N - \xi(\mathcal{O}_{N-2;N-2q-2})\cdot\alpha_{N-1}\beta_N. \end{aligned} \tag{6.20}$$

Hence for $1 \leqslant q \leqslant \frac{N}{2}-2$ the coefficient of λ^q in (6.19) is $\xi(\mathcal{O}_{N;N-2q})$. Similarly, we can show that the coefficient of λ^q is $\xi(\mathcal{O}_{N;N-2q})$ for $q = 0, \frac{N}{2}-1$.

When N is odd, we have (6.18) in a similar manner.

Finally, (6.18) holds for all $N \geqslant 2$ by induction. □

Lemma 6.3.

$$B(2,N-1) = \begin{cases} \sum_{q=0}^{\frac{N}{2}-2}\xi(\mathcal{E}_{N-1;N-2q-2})\cdot\lambda^q + \lambda^{\frac{N}{2}-1} & (N: \mathit{even}) \\ \sum_{q=0}^{\frac{N-1}{2}-1}\xi(\mathcal{E}_{N-1;N-2q-2})\cdot\lambda^q & (N: \mathit{odd}). \end{cases} \tag{6.21}$$

Proof. $B(2, N-1)$ is obtained from $B(1, N-2)$ by shifting all subscripts of elements by one. Therefore, by

$$B(1,N-2) = \begin{cases} \sum_{q=0}^{\frac{N}{2}-2}\xi(\mathcal{O}_{N-2;N-2q-2})\cdot\lambda^q + \lambda^{\frac{N}{2}-1} & (N: \text{even}) \\ \sum_{q=0}^{\frac{N-1}{2}-1}\xi(\mathcal{O}_{N-2;N-2q-2})\cdot\lambda^q & (N: \text{odd}) \end{cases}$$

we obtain (6.21). □

Using these results, we obtain the following lemma for B_N.

Lemma 6.4.

$$B_N = \begin{cases} \sum_{q=0}^{\frac{N}{2}-1}\xi(\mathcal{P}_{N;N-2q})\cdot\lambda^q + 2\lambda^{\frac{N}{2}} & (N: \mathit{even}) \\ \sum_{q=0}^{\frac{N-1}{2}}\xi(\mathcal{P}_{N;N-2q})\cdot\lambda^q & (N: \mathit{odd}) \end{cases} \tag{6.22}$$

Proof. By (6.5), (6.18) and (6.21), if N is even then

$$\begin{aligned} B_N &= \xi(\mathcal{O}_{N;N}) - \alpha_N\beta_1\cdot\xi(\mathcal{E}_{N-1;N-2}) + \sum_{q=1}^{\frac{N}{2}-2}\{\xi(\mathcal{O}_{N;N-2q}) + \xi(\mathcal{E}_{N-1;N-2q}) \\ &\quad - \alpha_N\beta_1\cdot\xi(\mathcal{E}_{N-1;N-2q-2})\}\lambda^q + \{\xi(\mathcal{O}_{N;2}) + \xi(\mathcal{E}_{N-1;2}) - \alpha_N\beta_1\}\lambda^{\frac{N}{2}-1} + 2\lambda^{\frac{N}{2}}. \end{aligned} \tag{6.23}$$

Since $\mathcal{P}_{N;N-2q}$ for $1 \leqslant q \leqslant \frac{N}{2}-2$ is decomposed as

$$\begin{aligned} \mathcal{P}_{N;N-2q} &= (\mathcal{O}_{N;N-2q}\backslash\{(\beta_1, x_2, \ldots, x_{N-1}, \alpha_N) \mid (1, x_2, \ldots, x_{N-1}) \in \mathcal{E}_{N-1;N-2q-2}\}) \\ &\quad \sqcup \mathcal{E}_{N;N-2q} \end{aligned}$$

we find

$$\xi(\mathcal{P}_{N;N-2q}) = \xi(\mathcal{O}_{N;N-2q}) + \xi(\mathcal{E}_{N;N-2q}) - \beta_1\cdot\xi(\mathcal{E}_{N-1;N-2q-2})\cdot\alpha_N. \tag{6.24}$$

Since

$$\mathcal{E}_{N;N-2q} = \{(x_1, x_2, \ldots, x_{N-1}, 1) \mid (x_1, x_2, \ldots, x_{N-1}) \in \mathcal{E}_{N-1;N-2q}\}$$

we have

$$\xi(\mathcal{E}_{N;N-2q}) = \xi(\mathcal{E}_{N-1;N-2q}).$$

Hence, the rhs of (6.24) coincides with the coefficient of λ^q and we have shown that for $1 \leqslant q \leqslant \frac{N}{2}-2$ the coefficient of λ^q in (6.23) is $\xi(\mathcal{P}_{N;N-2q})$. In a similar way, we can show that the coefficient of λ^q is $\xi(\mathcal{P}_{N;N-2q})$ for $q = 0, \frac{N}{2}-1$ respectively.

In the case that N is odd, we have (6.22) in a similar manner. □

Rewriting lemma 6.4 in terms of $1/K_n$ and $1/\widetilde{U}_n$ (instead of α_n and β_{n+1}) immediately gives theorem 6.1.

7. Conserved quantities of the PBBS for $M = 1$

In this section, we investigate the conserved quantities of the PBBS for $M = 1$ constructed from the conserved quantities of the ndKP equation.

From (6.4) and (6.11), we have

$$\begin{aligned} &\det(\lambda I + \widehat{L}(1; s)) \\ &= \begin{cases} \prod_{n=1}^N \left(\lambda - \frac{1}{\Theta_n}\right) + (-1)^{N+1}\eta\cdot\left(\sum_{k=0}^{\frac{N}{2}-1} f_k\lambda^k + 2\lambda^{\frac{N}{2}}\right)\Delta + (-1)^N\eta^2\Delta^2 & (N : \text{even}) \\ \prod_{n=1}^N \left(\lambda - \frac{1}{\Theta_n}\right) + (-1)^{N+1}\eta\cdot\left(\sum_{k=0}^{\frac{N-1}{2}} f_k\lambda^k\right)\Delta + (-1)^N\eta^2\Delta^2 & (N : \text{odd}). \end{cases} \end{aligned} \tag{7.1}$$

By expanding (7.1) in terms of λ, we find

(i) if N is even,

$$e_k = \begin{cases} (-1)^N\frac{1}{\Theta_1\cdots\Theta_N} + (-1)^N\eta^2\Delta^2 & \\ \quad + (-1)^{N+1}\eta\cdot\left(\frac{1}{\widetilde{U}_1\cdots\widetilde{U}_N} + \frac{1}{K_1\cdots K_N}\right)\Delta & (k = 0) \\ (-1)^{\bar{k}}\sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N}\frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}} + (-1)^{N+1}\eta\cdot f_k\Delta & (0 < k < N/2, k \in \mathbb{Z}) \\ (-1)^{\bar{k}}\sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N}\frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}} + (-1)^{N+1}2\eta\cdot\Delta & (k = N/2) \\ (-1)^{\bar{k}}\sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N}\frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}} & (N/2 < k \leqslant N, k \in \mathbb{Z}) \end{cases}$$

(ii) if N is odd

$$e_k = \begin{cases} (-1)^N \frac{1}{\Theta_1\cdots\Theta_N} + (-1)^N \eta^2 \Delta^2 \\ \quad + (-1)^{N+1}\eta \cdot \left(\frac{1}{\widetilde{U}_1\cdots\widetilde{U}_N} + \frac{1}{K_1\cdots K_N}\right)\Delta & (k=0) \\ (-1)^{\bar{k}} \sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N} \frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}} + (-1)^{N+1}\eta\cdot f_k\Delta & (0<k<N/2, k\in\mathbb{Z}) \\ (-1)^{\bar{k}} \sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N} \frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}} & (N/2<k\leqslant N, k\in\mathbb{Z}) \end{cases}$$

where $\bar{k} := N-k$.

Now we consider the ultradiscrete limit of e_k. Since

$$\begin{aligned} -\lim_{\varepsilon\to+0} \varepsilon \log\left(\sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N} \frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}}\right) &= -\lim_{\varepsilon\to+0}\varepsilon \\ &\times \log \sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N} \exp\big(-\big(\theta_{n_1}+\cdots+\theta_{n_{\bar{k}}}\big)/\epsilon\big)\big(1+\exp\big(-(\theta_{n_1}+\cdots+\theta_{n_{\bar{k}}})/\epsilon\big)\big) \\ &= -\max\big\{-\big(\theta_{n_1}+\cdots+\theta_{n_{\bar{k}}}\big) \mid 1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N\big\} \\ &= \min\big\{\theta_{n_1}+\cdots+\theta_{n_{\bar{k}}} \,\big|\, 1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N\big\}. \end{aligned} \tag{7.2}$$

and θ_n is the capacity of the box, the ultradiscrete limits of e_k give trivial conserved quantities for $N/2 \leqslant k \leqslant N, k\in\mathbb{Z}$. We are not interested in these.

Let $e_k^{[i]}$ be the coefficient of η^i in e_k. As mentioned before, η is an independent parameter and, therefore, $e_k^{[i]}$ itself is conserved in time. When $0<k<N/2, k\in\mathbb{Z}$, the ultradiscrete limits corresponding to $e_k^{[0]}$ and $e_k^{[1]}$ are given by

$$\begin{aligned} ue_k^{[0]} &:= -\lim_{\varepsilon\to+0}\varepsilon\log(-1)^{\bar{k}} e_k^{[0]} \\ &= -\lim_{\varepsilon\to+0}\varepsilon\log\left(\sum_{1\leqslant n_1<\cdots<n_{\bar{k}}\leqslant N}\frac{1}{\Theta_{n_1}\cdots\Theta_{n_{\bar{k}}}}\right) \end{aligned} \tag{7.3}$$

$$\begin{aligned} ue_k^{[1]} &:= -\lim_{\varepsilon\to+0}\varepsilon\log(-1)^{N+1} e_k^{[1]} \\ &= -\lim_{\varepsilon\to+0}\varepsilon\log f_k\Delta. \end{aligned} \tag{7.4}$$

The conserved quantity $ue_k^{[0]}$ is trivial, and we will therefore only pay attention to $ue_k^{[1]}$. According to theorem 6.1, we define the set

$$F_k := \big\{x_1+x_2+\cdots+x_N \,\big|\, (x_1,x_2,\ldots,x_N)\in\mathcal{K}_{N;N-2k}^{(\kappa_n,u_n,0)}\big\} \tag{7.5}$$

where

$$\mathcal{K}_{N;N-2k}^{(\kappa_n,u_n,0)} \equiv \mathcal{K}_{N;N-2k}\,|_{1/K_n\to\kappa_n, 1/\widetilde{U}_n\to u_n, 1\to 0}.$$

Since

$$\begin{aligned} \lim_{\varepsilon\to+0}\varepsilon\log\Delta = \lim_{\varepsilon\to+0}\varepsilon\log\prod_{j=1}^{N}(1+\mathrm{e}^{-\theta_j/\epsilon}) = \sum_{j=1}^{N}\max\{0,-\theta_j\} \\ = 0 \end{aligned}$$

and $\lim_{\varepsilon\to+0}\varepsilon\log\widetilde{U}_n = u_n$, $ue_k^{[1]}$ is given by

$$ue_k^{[1]} = \min F_k. \tag{7.6}$$

Here $\min F_k$ denotes the minimum element in the set F_k.

In the case $k=0$, the ultradiscrete limits of $e_0^{[i]}$ $(i=0,1,2)$ do not give nontrivial conserved quantities.

Next we will show that $ue_k^{[1]}$ (7.6) coincides with the conserved quantities given in [13] when ${}^{\forall}n\,\theta_n = 1$, i.e., all the boxes have capacity one. Our aim is to prove theorem 7.1. For this purpose we have to prepare several lemmas and propositions.

We denote by $\mathbf{p}$ a 01 sequence corresponding to a state of the PBBS. Due to the periodic boundary condition of the PBBS, the last entry of $\mathbf{p}$ is regarded as being adjacent to the first one. We assume that the state has n_p *solitons*, meaning that $\mathbf{p}$ contains n_p sequences of '1' or equivalently the same number of sequences of '0'.

We also consider a sequence of 'b', 'w' and 'ϕ' and call it a $bw\phi$ sequence. In a $bw\phi$ sequence, the last entry should be regarded as being adjacent to the first entry too. As shown below, the letters 'b', 'w' and 'ϕ' correspond to u_n, κ_n and 0 respectively.

Let $I_{N;N-2k}$ be the set of $bw\phi$ sequences of length N defined by

$$I_{N;N-2k} := \left\{x_1x_2\cdots x_N \,\middle|\, (x_1, x_2, \ldots, x_N) \in \mathcal{K}^{(w,b,\phi)}_{N;N-2k}\right\}$$

where

$$\mathcal{K}^{(w,b,\phi)}_{N;N-2k} \equiv \mathcal{K}_{N;N-2k}\big|_{1/K_n\to w,1/\widetilde{U}_n\to b,1\to\phi}.$$

Note that a $bw\phi$ sequence $\in I_{N;N-2k}$ contains $2k$ 'ϕ' (hence it contains $N-2k$ 'b' and 'w') and that there are neither consecutive sequences bw nor wb.

Suppose that p_n ($p_n \in \{0, 1\}$) is the nth element of $\mathbf{p}$ of length N and q_n ($q_n \in \{b, w, \phi\}$) is the nth element of a $bw\phi$ sequence $\mathbf{b} \in I_{N;N-2k}$. Then we define $g_{\mathbf{p}}(\mathbf{b})$ by

$$g_{\mathbf{p}}(\mathbf{b}) := \sum_{n=1}^{N}[p_n, q_n] \tag{7.7}$$

where $[\cdots,\cdots]$ is the map:

$$\{0, 1\} \times \{b, w, \phi\} \;\to\; \{0, 1\}$$

given by

$$\begin{array}{lll} [0, b] = 0 & [0, w] = 1 & [0, \phi] = 0 \\ [1, b] = 1 & [1, w] = 0 & [1, \phi] = 0. \end{array}$$

Remark 7.1. If we identify u_n, κ_n and 0 with 'b', 'w' and 'ϕ' respectively, we find that $ue_k^{[1]}$ for a state $\mathbf{p}$ is given as

$$ue_k^{[1]} = \min F_k = \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}). \tag{7.8}$$

A $bw\phi$ sequence is composed of consecutive 'b', 'w' and 'ϕ'. We shall call such a disjoint sequence of one kind of letter *a band*. We sometimes write a b-band (w-band, ϕ-band) instead of a band of letters 'b', 'w', 'ϕ'.

Example 7.1. For $N = 22$, $k = 5$, a $bw\phi$-sequence $\mathbf{b} \in I_{N;N-2k}$

$$\mathbf{b} = bb\phi www\phi\phi www\phi\phi\phi bb\phi\phi\phi\phi bb$$

consists of eight bands: '$bbbb$', 'ϕ', 'www', '$\phi\phi$', 'www', '$\phi\phi\phi$', 'bb' and '$\phi\phi\phi\phi$'. (Recall that the last entry is adjacent to the first one.) Note that the number of 'ϕ' is $2k = 10$.

Using the notion of bands, we define the sets $M_k^{(N)}$ ($k = 1, 2, \ldots, N$) as

$$M_k^{(N)} := \left\{ x_1x_2\cdots x_N \,\middle|\, \begin{array}{l} x_i \in \{b, w\}\ (i = 1, 2, \ldots, N) \\ x_1x_2\cdots x_N \text{ consists of} \\ k\ b\text{-bands and } k\ w\text{-bands.} \end{array} \right\}.$$

We also define $M_0^{(N)} := \{bb\cdots b, ww\cdots w\}$. Note that the elements of $M_k^{(N)}$ do not contain a letter 'ϕ'. Hence the number of bands of 'b' is equal to that of 'w'.

For a given $\mathbf{a} \in M_k^{(N)}$, there are $2k$ boundaries between the bands. We denote by $\mathbf{a}^{\langle +\phi\rangle}$ the $bw\phi$ sequence which is constructed from $\mathbf{a}$ by replacing each of the $2k$ letters at the right of the boundaries with 'ϕ'.

Example 7.2. For $N = 20$ and $k = 4$,

$$\mathbf{a} = bbbbwwwbwbwbbwwwbbbb \in M_4$$

has eight boundaries. It yields

$$\mathbf{a}^{\langle +\phi\rangle} = bbbb\phi ww\phi\phi\phi\phi\phi b\phi ww\phi bbb \in I_{N;N-2k} = I_{20,12}.$$

Since the number of boundaries in $\mathbf{a}$ is $2k$ and an odd number of 'ϕ' is inserted between the bands in $\mathbf{a}^{\langle +\phi\rangle}$, we have the following lemma 7.1.

Lemma 7.1. $\mathbf{a}^{\langle +\phi\rangle} \in I_{N;N-2k}$ *for any* $\mathbf{a} \in M_k^{(N)}$.

Proposition 7.1.

$$\min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) \geqslant \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b})$$

Proof. By the definition of $g_{\mathbf{p}}$ (7.7), for any $\mathbf{a} \in M_k^{(N)}$, we find

$$g_{\mathbf{p}}(\mathbf{a}) \geqslant g_{\mathbf{p}}(\mathbf{a}^{\langle +\phi\rangle}).$$

However, from lemma 7.1, $\mathbf{a}^{\langle +\phi\rangle} \in I_{N;N-2k}$ and the proposition is proved. □

Lemma 7.2. *Let* $\mathbf{b}^*$ *be the sequence which is obtained from a sequence* $\mathbf{b} \in I_{N;N-2k}$ *by replacing all the letters '*ϕ*' with '*b*' or '*w*' arbitrarily. Then*

$$\mathbf{b}^* \in \sqcup_{i=0}^{k} M_i^{(N)}.$$

Proof. By the definition of $I_{N;N-2k}$, no b-band in $\mathbf{b}$ can be adjacent to a w-band. If a ϕ-band is in between two b-bands or two w-bands, it contains an even number of 'ϕ'. On the other hand, if a ϕ-band is in between a b-band and a w-band, it contains an odd number of 'ϕ'. Hence, by changing a 'ϕ' to a 'b' or a 'w', we can make the number of boundaries in $\mathbf{b}^*$ at most equal to that of 'ϕ'. Since $\mathbf{b} \in I_{N;N-2k}$, it contains $2k$ 'ϕ'. Hence $\mathbf{b}^*$ has at most k bands of 'b'. □

Lemma 7.3.

$$\min_{\mathbf{a}\in \sqcup_{i=0}^{k} M_i^{(N)}} g_{\mathbf{p}}(\mathbf{a}) \leqslant \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b})$$

Proof. Let $\mathbf{p} = (p_1, p_2, \ldots, p_N)$. For a $\mathbf{b} := (q_1, q_2, \ldots, q_N) \in I_{N;N-2k}$, we define $\mathbf{b}^{\langle -\phi\rangle} := (q_1', q_2', \ldots, q_N')$ as follows. If $q_i \neq \phi$, $q_i' = q_i$. If $q_i = \phi$ and $p_i = 1$, $q_i' = b$, and if $q_i = \phi$ and $p_i = 0$, $q_i' = w$. Then, by the definition of g, $g_{\mathbf{p}}(\mathbf{b}) = g_{\mathbf{p}}(\mathbf{b}^{\langle -\phi\rangle})$. Furthermore, from lemma 7.2, $\mathbf{b}^{\langle -\phi\rangle} \in \sqcup_{i=0}^{k} M_i^{(N)}$. Hence

$$\begin{aligned}\min_{\mathbf{a}\in \sqcup_{i=0}^{k} M_i^{(N)}} g_{\mathbf{p}}(\mathbf{a}) &\leqslant \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}^{\langle -\phi\rangle}) \\ &= \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}).\end{aligned}$$

□

Proposition 7.2. *Let n_p be the number of blocks of consecutive '1' in* $\mathbf{p}$*. Then*

$$\begin{cases} \min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) < \min_{\mathbf{a}\in M_{k-1}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) & \textit{for} \quad n_p \geqslant k \\ \min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) > \min_{\mathbf{a}\in M_{n_p}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) & \textit{for} \quad n_p < k. \end{cases} \tag{7.9}$$

Note that n_p is the number of solitons in the PBBS corresponding to $\mathbf{p}$.

Proof. Suppose that $\mathbf{a}'_* = (q_1, q_2, \ldots, q_N) \in M_{k-1}^{(N)}$ is the sequence which attains $g_{\mathbf{p}}(\mathbf{a}'_*) = \min_{\mathbf{a}\in M_{k-1}^{(N)}} g_{\mathbf{p}}(\mathbf{a})$. For $n_p \geqslant k$, there exists an entry p_n in $\mathbf{p}$ which satisfies $[q_n, p_n] = 1$. Let $\mathbf{a}' = (q'_1, q'_2, \ldots, q'_N)$ be the sequence obtained from $\mathbf{a}'_*$ by changing some of the q_n. Clearly $g_{\mathbf{p}}(\mathbf{a}') < g_{\mathbf{p}}(\mathbf{a}'_*)$. Furthermore: $\mathbf{a}'$ belongs to $M_k^{(N)}$. Because $\mathbf{a}'$ belongs to $M_{k-1}^{(N)} \sqcup M_k^{(N)}$, by construction, and $\mathbf{a} \notin M_{k-1}^{(N)}$ due to the definition of $\mathbf{a}'_*$, we have

$$\min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) < \min_{\mathbf{a}\in M_{k-1}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) \qquad \text{for} \quad n_p \geqslant k.$$

Since $\mathbf{p}$ has n_p sequences of consecutive '1' (and '0'), if we define

$$\mathbf{a}_0 := \mathbf{p}|_{0\to b, 1\to w}$$

we have that $\mathbf{a}_0 \in M_{n_p}^{(N)}$ and $g_{\mathbf{p}}(\mathbf{a}_0) = 0$. Since $\mathbf{a}_0$ is the only $bw\phi$ sequence that does not contain 'ϕ' and gives $g_{\mathbf{p}} = 0$, we have

$$\min_{\mathbf{a}\in M_{n_p}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) < \min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) \qquad \text{for} \quad k > n_p.$$

□

The following two corollaries are a direct consequence of this proposition and lemma 7.3.

Corollary 7.1. *For $k \leqslant n_p$,*

$$\min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) = \min_{\mathbf{a}\in \sqcup_{i=0}^k M_i^{(N)}} g_{\mathbf{p}}(\mathbf{a}).$$

Corollary 7.2. *For $k \leqslant n_p$,*

$$\min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) \leqslant \min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}).$$

Proposition 7.3. *For $k \leqslant n_p$,*

$$\min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}) = \min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}).$$

For $k \geqslant n_p$,

$$\min_{\mathbf{b}\in I_{N;N-2k}} g_{\mathbf{p}}(\mathbf{b}) = \min_{\mathbf{a}\in M_{n_p}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) = 0.$$

Proof. The former part follows from proposition 7.1 and corollary 7.2. For the latter part, we consider $\mathbf{a}_0 \in M_{n_p}^{(N)}$ in the proof of proposition 7.2. By changing some letters in $\mathbf{a}_0$ into 'ϕ', we obtain a sequence belonging to $I_{N;N-2k}$. Clearly it gives $g_{\mathbf{p}} = 0$. □

Remark 7.2. From (7.8), we have that for a 01 sequence $\mathbf{p}$:

$$ue_k^{[1]} = \begin{cases} \min_{\mathbf{a}\in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}) & \text{for} \quad k \leqslant n_p, \\ \min_{\mathbf{a}\in M_{n_p}^{(N)}} g_{\mathbf{p}}(\mathbf{a}) & \text{for} \quad k > n_p. \end{cases} \tag{7.10}$$

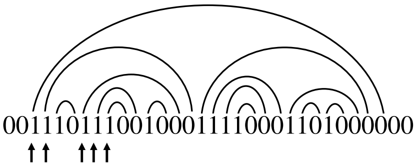

Figure 6. An example of a block. The '1' pointed at by vertical arrows are the components of the largest soliton in the block.

In the discussion below, we need the notion of *a block in* $\mathbf{p}$. Its definition and important properties were explained in detail in [13]. We briefly review its definition.

For a 01 sequence $\mathbf{p}$, we draw arc lines between 10 pairs which give the conserved quantities p_1 from the introduction. Then we draw arc lines between 10 pairs for p_2 over the arc lines drawn previously. We repeat this procedure until p_s arc lines have been drawn for the last 10 pairs over the other arc lines. Then we find several boundaries between '0' and '1' or '0' and '0' over which there is no arc line (see figure 6). A block in $\mathbf{p}$ is a 01 sequence which is located between two successive boundaries, all the entries of which are connected by arc lines.

Lemma 7.4. *Suppose that* $\mathbf{a}_* = (q_1^*, q_2^*, \ldots, q_N^*) \in M_k^{(N)}$ $(k \leqslant n_p)$ *satisfies*

$$g_{\mathbf{p}}(\mathbf{a}_*) = \min_{\mathbf{a} \in M_k^{(N)}} g_{\mathbf{p}}(\mathbf{a}).$$

If the nth entry of $\mathbf{p}$, p_n, *does not belong to a block,* $p_n = 0$ *and* $q_n^* = b$.

Proof. By the definition of a block, an entry is '0' when it does not belong to a block. Hence p_n must be '0' and is in a sequence of consecutive '0' between two blocks.

Suppose that $q_n^* = w$. We denote the w-band with q_n^* by $\boldsymbol{w}$, and the corresponding sequence in $\mathbf{p}$ by $\mathbf{p}_w$. If the right edge of $\mathbf{p}_w$ belongs to a block, we define $\mathbf{p}_w^* \subset \mathbf{p}_w$ as the sequence obtained from $\mathbf{p}_w$ by eliminating the part belonging to the block. Otherwise we put $\mathbf{p}_w^* = \mathbf{p}_w$. The sequence $\mathbf{p}_w^*$ always has more '0' than '1', because a block contains the same number of '0' and '1', and, when we cut a block into two sequences, the right sequence always has more '0' than '1'.

Now we define by $\mathbf{a}_{**}$ the sequence obtained from $\mathbf{a}_*$ by replacing all the 'w' corresponding to $\mathbf{p}_w^*$ with 'b'. Clearly $\mathbf{a}_{**} \in M_k^{(N)} \sqcup M_{k-1}^{(N)}$ and

$$g_{\mathbf{p}}(\mathbf{a}_{**}) < g_{\mathbf{p}}(\mathbf{a}_*).$$

By the definition of $\mathbf{a}_*, \mathbf{a}_{**} \notin M_k^{(N)}$ and $\mathbf{a}_{**} \in M_{k-1}^{(N)}$. But this contradicts proposition 7.2. Therefore the assumption $q_n^* = w$ is wrong and $q_n^* = b$. □

Hereafter we explicitly introduce the system size (N) dependence of $\mathbf{p}$, $\mathbf{a}$, etc as $\mathbf{p}^{(N)}$, $\mathbf{a}^{(N)}$, etc.

Proposition 7.4. *Let* $\mathbf{p}^{(n)} = p_1 p_2 \cdots p_n$. *There is at least one entry* p_i $(i \in \{1, 2, \ldots, n\})$ *which does not belong to a block. We define* $\mathbf{p}^{(n)}(i; j) := p_1 p_2 \cdots p_i \underbrace{00 \cdots 0}_{j} p_{i+1} \cdots p_n$. *Accordingly, for a sequence* $\mathbf{a}_*^{(n)} := a_1 a_2 \cdots a_n \in M_k^{(N)}$, *we define the new sequence* $\mathbf{a}_*^{(n)}(i; j)$ *by* $\mathbf{a}_*^{(n)}(i; j) := a_1 a_2 \cdots a_i \underbrace{bb \cdots b}_{j} a_{i+1} \ldots a_n$. *If* $g_{\mathbf{p}^{(n)}}(\mathbf{a}_*^{(n)}) = \min_{\mathbf{a}^{(n)} \in M_k^{(n)}} g_{\mathbf{p}^{(n)}}(\mathbf{a}^{(n)})$ *then*

$$g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_*^{(n)}(i; j)) = \min_{\mathbf{a}^{(n+j)} \in M_k^{(n+j)}} g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}^{(n+j)}).$$

Proof. Let $\mathbf{a}_{**}^{(n+j)} \in M_k^{(n+j)}$ be the sequence which minimizes $g_{\mathbf{p}^{(n)}(i;j)}$, i.e.,

$$g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{**}^{(n+j)}) = \min_{\mathbf{a}^{(n+j)} \in M_k^{(n+j)}} g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}^{(n+j)}).$$

From lemma 7.4, $\mathbf{a}_{**}^{(n+j)}$ is expressed as $\mathbf{a}_{**}^{(n+j)} = a'_1 a'_2 \cdots a'_i \underbrace{bb \cdots b}_{j} a'_{i+1} \dots a'_n$. (Note that $a'_i = b$.) If we define $\bar{\mathbf{a}}_{**}^{(n)} := a'_1 a'_2 \cdots a'_i a'_{i+1} \dots a'_n$, we find that

$$g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{**}^{(n+j)}) = g_{\mathbf{p}^{(n)}}(\bar{\mathbf{a}}_{**}^{(n)}) \geqslant g_{\mathbf{p}^{(n)}}(\mathbf{a}_{*}^{(n)}).$$

However, by the definition of $\mathbf{a}_*^{(n)}(i;\, j)$,

$$g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{**}^{(n+j)}) \leqslant g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{*}^{(n)}(i;\, j)) = g_{\mathbf{p}^{(n)}}(\mathbf{a}_{*}^{(n)}).$$

Therefore $g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{*}^{(n)}(i;\, j)) = g_{\mathbf{p}^{(n)}(i;j)}(\mathbf{a}_{**}^{(n+j)})$.

□

Lemma 7.5. *For $k \leqslant n_p$, we assume that $\mathbf{a}_*^{(N)} \in M_k^{(N)}$ satisfies*

$$g_{\mathbf{p}^{(N)}}(\mathbf{a}_*^{(N)}) = \min_{\mathbf{a}^{(N)} \in M_k^{(N)}} g_{\mathbf{p}^{(N)}}(\mathbf{a}^{(N)}).$$

Suppose that the 01-sequence $\mathbf{p}^{(N)}$ has evolved into $\mathbf{p}_{(T)}^{(N)}$ after T time steps according to the time evolution rule for the PBBS. Then

$$g_{\mathbf{p}_{(T)}^{(N)}}\left(\mathbf{a}_{*(T)}^{(N)}\right) = g_{\mathbf{p}^{(N)}}(\mathbf{a}_*^{(N)}).$$

where $\mathbf{a}_{*(T)}^{(N)} \in M_k^{(N)}$ denotes the sequence which satisfies

$$g_{\mathbf{p}_{(T)}^{(N)}}\left(\mathbf{a}_{*(T)}^{(N)}\right) = \min_{\mathbf{a}^{(N)} \in M_k^{(N)}} g_{\mathbf{p}_{(T)}^{(N)}}(\mathbf{a}^{(N)}).$$

Proof. From (7.10), $g_{\mathbf{p}^{(N)}}(\mathbf{a}_*^{(N)})$ is a conserved quantity in time. □

The following proposition is immediately obtained from proposition 7.3 and lemma 7.5.

Proposition 7.5. *Let $\mathbf{p}^{(N)}$, $\mathbf{a}_*^{(N)}$, $\mathbf{p}_{(T)}^{(N+j)}$ and $\mathbf{a}_{*(T)}^{(N+j)}$ be the sequences given in proposition 7.3 and lemma 7.5. Then we have*

$$g_{\mathbf{p}^{(N)}}(\mathbf{a}_*^{(N)}) = g_{\mathbf{p}_{(T)}^{(N+j)}}\left(\mathbf{a}_{*(T)}^{(N+j)}\right)$$

Lemma 7.6 ([13])**.** *Let n_p be the number of solitons in $\mathbf{p}^{(N)}$. We denote their lengths by $L_1, L_2, \dots, L_{n_p}$ $\left(L_1 \geqslant L_2 \geqslant \cdots \geqslant L_{n_p}\right)$. Then, for sufficiently large j, there exist time steps T such that $\mathbf{p}_{(T)}^{(N+j)}$ satisfies the following conditions.*

1. *$\mathbf{p}_{(T)}^{(N+j)}$ consists of s bands of consecutive '1' and the same number of bands of '0'.*
2. *The length of the ith band of '1' is L_i $(i = 1, 2, \dots, n_p)$.*

Theorem 7.1. *Let S be the number of '1' in $\mathbf{p}^{(N)}$. For a $\mathbf{p}^{(N)}$ with n_p solitons,*

$$ue_k^{[1]} = S - \sum_{i=1}^{k} L_i \qquad \text{for} \quad k \leqslant n_p \tag{7.11}$$

and $ue_k^{[1]} = 0$ for $k \geqslant n_p$. Here L_i is the length of the ith soliton.

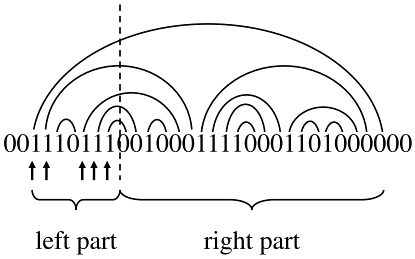

Figure 7. The left part and the right part of the block given in figure 6. Two smaller blocks are left when we eliminate the largest arc which connects the 10 pair at the edges of the block.

Proof. We assume that j and T are the positive integers given in proposition 7.5. Let $\mathbf{a}_{*(T)}^{(N+j)} \in M_k^{(N+j)}$ be the sequence which satisfies

$$g_{\mathbf{p}_{(T)}^{(N+j)}}\left(\mathbf{a}_{*(T)}^{(N+j)}\right) = \min_{\mathbf{a}^{(N+j)} \in M_k^{(N+j)}} g_{\mathbf{p}_{(T)}^{(N+j)}}(\mathbf{a}^{(N+j)}).$$

From lemma 7.6, there are n_p bands of '1' with lengths $L_1, L_2, \ldots, L_{n_p}$ $\left(L_1 \geqslant L_2 \geqslant \cdots \geqslant L_{n_p}\right)$. Since $[w, 1] = 0$ and $[b, 1] = 1$, k w-bands must correspond to the sequences of '1' with lengths $L_1, L_2, \ldots, L_k$ for $k \leqslant n_p$. Hence we find

$$g_{\mathbf{p}_{(T)}^{(N+j)}}\left(\mathbf{a}_{*(T)}^{(N+j)}\right) = S - \sum_{i=1}^{k} L_i.$$

Then proposition 7.5 and (7.10) prove the theorem. □

Finally we give a method which can be used to construct $\mathbf{a}_*^{(N)} \in M_k$ which minimizes $g_{\mathbf{p}^{(N)}}$ for a $\mathbf{p}^{(N)}$. For this purpose, we need some properties of blocks.

Definition 7.1 ([13]). *We divide a block into two parts. Let p be the position of the rightmost '1' which belongs to the largest soliton in the block. The right part of the block is the 01 sequence which is located on the right-hand side of p. (The '1' at p does not belong to the right part.) The remainder is called the left part of the block.*

Lemma 7.7 ([13]).

1. *A block contains solitons. When the length of the largest soliton in it is L, the left part of the block has L more '1' than '0', and the right part has L more '0' than '1'.*
2. *The edges of both the right part and the left part of a block belong to 10 pairs whose '1' constitute the largest soliton in the block.*
3. *If we eliminate these 10 pairs, the remainders constitute disjoint blocks in the left part and the right part respectively.*

We put $\mathbf{p}^{(N)} := (p_1, p_2, \ldots, p_N)$ and assume that it has n_p solitons. Each soliton constitutes a block. We denote the block of the ith largest soliton by $\mathbf{p}_i$ $(\subset \mathbf{p}^{(N)})$. Now we define sequences of letters 'b' and 'w', $\mathbf{a}_{(i)} \in M_i^{(N)}$ $(i = 0, 1, 2, \ldots, k)$ $(k \leqslant N)$ as follows.

1. $\mathbf{a}_{(0)} = \underbrace{bbb\cdots b}_{N}$.
2. We replace the 'b' in the part of $\mathbf{a}_{(0)}$ which corresponds to the left part of $\mathbf{p}_1$ with 'w'. We denote the new sequence by $\mathbf{a}_{(1)}$.

3. From lemma 7.7, we see the part of $\mathbf{a}_{(1)}$ which corresponds to $\mathbf{p}_2$ consists of either only 'b' or only 'w'. If it consists only of 'b', then we replace the 'b' which corresponds to the left part of $\mathbf{p}_2$ with 'w'. Otherwise we replace the 'w' which corresponds to the right part of $\mathbf{p}_2$ with 'b'. We denote the new sequence by $\mathbf{a}_{(2)}$.
4. Repeat the above procedure to obtain $\mathbf{a}_{(i+1)}$ from $\mathbf{a}_{(i)}$ and $\mathbf{p}_{i+1}$ $(i = 2, 3, \ldots, k-1)$.

We then have the following result for $\mathbf{a}_*^{(N)}$:

Proposition 7.6. *In the above notation, we have* $\mathbf{a}_{(k)} = \mathbf{a}_*^{(N)}$.

Proof. From lemma 7.7, we easily find $\mathbf{a}_{(i)} \in M_i^{(N)}$ and $g_{\mathbf{p}^{(N)}}(\mathbf{a}_{(i)}) = S - \sum_{j=1}^{i} L_j$. □

8. Concluding remarks

In this paper, we showed that the generalized PBBSs are obtained from a reduction of the ndKP equation through ultradiscretization. Using the Lax representation of the ndKP equation, we have shown a formula to calculate the conserved quantities of the PBBS and we gave an explicit form of the conserved quantities in the case of only one kind of ball. We also proved that these conserved quantities coincide with those obtained previously when all the box capacities are restricted to one.

For the simplest PBBS, the formula used to calculate the fundamental cycle is explicitly obtained using the conserved quantities and some rescaling properties of the states. The formula reveals important properties of the PBBS such as the integrable nature of the PBBS as a dynamical system, combinatorial features, and number theoretical aspects related to the Riemann hypothesis. To investigate similar properties for the generalized PBBS is one of the important future problems.

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